

# A STOCHASTIC DATKO-PAZY THEOREM

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**ABSTRACT.** Let  $H$  be a Hilbert space and  $E$  a Banach space. In this note we present a sufficient condition for an operator  $R : H \rightarrow E$  to be  $\gamma$ -radonifying in terms of Riesz sequences in  $H$ . This result is applied to recover a result of Lutz Weis and the second named author on the  $R$ -boundedness of resolvents, which is used to obtain a Datko-Pazy type theorem for the stochastic Cauchy problem. We also present some perturbation results.

**Subject classifications:** Primary: 47D06, 28C20, Secondary: 46B09, 46B15, 47N30

**Key words:** semigroups, Datko-Pazy theorem, stochastic Cauchy problem, invariant measures, perturbation theory Riesz sequences, almost summing operators,  $\gamma$ -radonifying operators

## 1. INTRODUCTION

The well-known Datko-Pazy theorem states that if  $(T(t))_{t \geq 0}$  is a strongly continuous semigroup on a Banach space  $E$  such that all orbits  $T(\cdot)x$  belong to the space  $L^p(\mathbb{R}_+, E)$  for some  $p \in [1, \infty)$ , then  $(T(t))_{t \geq 0}$  is uniformly exponentially stable, or equivalently, there exists an  $\varepsilon > 0$  such that all orbits  $t \mapsto e^{\varepsilon t}T(t)x$  belong to  $L^p(\mathbb{R}_+, E)$ . For  $p = 2$  and Hilbert spaces  $E$  this result is due to Datko [3], and the general case was obtained by Pazy [14].

In this note we prove a stochastic version of the Datko-Pazy theorem for spaces of  $\gamma$ -radonifying operators (cf. Section 2). Let us denote by  $\gamma(\mathbb{R}_+, E)$  the space of all strongly measurable functions  $\phi : \mathbb{R}_+ \rightarrow E$  for which the integral operator

$$f \mapsto \int_0^\infty f(t)\phi(t) dt$$

is well-defined and  $\gamma$ -radonifying from  $L^2(\mathbb{R}_+)$  to  $E$ .

**Theorem 1.1a** (Stochastic Datko-Pazy Theorem, first version). *Let  $A$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ . The following assertions are equivalent:*

- (a) *For all  $x \in E$ ,  $T(\cdot)x \in \gamma(\mathbb{R}_+, E)$ .*
- (b) *There exists an  $\varepsilon > 0$  such that for all  $x \in E$ ,  $t \mapsto e^{\varepsilon t}T(t)x \in \gamma(\mathbb{R}_+, E)$ .*

If  $E$  is a Hilbert space,  $\gamma(\mathbb{R}_+, E) = L^2(\mathbb{R}_+, E)$  and Theorem 1.1a is equivalent to the Datko's theorem mentioned above.

As explained in [12],  $\gamma$ -radonifying operators play an important role in the study of the following stochastic abstract Cauchy problem on  $E$ :

$$(\text{SCP})_{(A,B)} \quad \begin{cases} dU(t) &= AU(t) dt + B dW_H(t), & t \geq 0, \\ U(0) &= 0. \end{cases}$$

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*Date:* 2nd February 2008.

The authors gratefully acknowledge financial support by a 'VIDI subsidie' (639.032.201) in the 'Vernieuwingsimpuls' programme of the Netherlands Organization for Scientific Research (NWO). The second named author is also supported by a Research Training Network (HPRN-CT-2002-00281).

Here,  $H$  is a separable Hilbert space,  $B \in \mathcal{B}(H, E)$  is a bounded operator, and  $W_H$  is an  $H$ -cylindrical Brownian motion.

Theorem 1.1a can be reformulated in terms of invariant measures for  $(\text{SCP})_{(A,B)}$  as follows.

**Theorem 1.1b** (Stochastic Datko-Pazy theorem, second version). *With the above notations, the following assertions are equivalent:*

- (a) *For all rank one operators  $B \in \mathcal{B}(H, E)$ , the problem  $(\text{SCP})_{(A,B)}$  admits an invariant measure.*
- (b) *There exists an  $\varepsilon > 0$  such that for all rank one operators  $B \in \mathcal{B}(H, E)$ , the problem  $(\text{SCP})_{(A+\varepsilon, B)}$  admits an invariant measure.*

For unexplained terminology and more information on the stochastic Cauchy problem and invariant measures we refer to [2, 11, 12].

## 2. RIESZ BASES AND $\gamma$ -RADONIFYING OPERATORS

Let  $\mathcal{H}$  be a Hilbert space and  $E$  a Banach space. Let  $(\gamma_n)_{n \geq 1}$  be a sequence of independent standard Gaussian random variables on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . A bounded linear operator  $R : \mathcal{H} \rightarrow E$  is called *almost summing* if

$$\|R\|_{\gamma_\infty(\mathcal{H}, E)} := \sup \left\| \sum_{n=1}^N \gamma_n R h_n \right\|_{L^2(\Omega, E)} < \infty,$$

where the supremum is taken over all  $N \in \mathbb{N}$  and all orthonormal systems  $\{h_1, \dots, h_N\}$  in  $\mathcal{H}$ . Endowed with this norm, the space  $\gamma_\infty(\mathcal{H}, E)$  of all almost summing operators is a Banach space. Moreover,  $\gamma_\infty(\mathcal{H}, E)$  is an operator ideal in  $\mathcal{B}(\mathcal{H}, E)$ . The closure of the finite rank operators in  $\gamma_\infty(\mathcal{H}, E)$  will be denoted by  $\gamma(\mathcal{H}, E)$ . Operators belonging to this space are called  *$\gamma$ -radonifying*. Again  $\gamma(\mathcal{H}, E)$  is an operator ideal in  $\mathcal{B}(\mathcal{H}, E)$ .

Let us now assume that  $\mathcal{H}$  is a separable Hilbert space. Under this assumption one has  $R \in \gamma_\infty(\mathcal{H}, E)$  if and only if for some (every) orthonormal basis  $(h_n)_{n \geq 1}$  for  $\mathcal{H}$ ,

$$M := \sup_{N \geq 1} \left\| \sum_{n=1}^N \gamma_n R h_n \right\|_{L^2(\Omega, E)} < \infty.$$

In that case,  $\|R\|_{\gamma_\infty(\mathcal{H}, E)} = M$ . Furthermore, one has  $R \in \gamma(\mathcal{H}, E)$  if and only if for some (every) orthonormal basis  $(h_n)_{n \geq 1}$  for  $\mathcal{H}$ ,  $\sum_{n \geq 1} \gamma_n R h_n$  converges in  $L^2(\Omega, E)$ . In that case,

$$\|R\|_{\gamma(\mathcal{H}, E)} = \left\| \sum_{n \geq 1} \gamma_n R h_n \right\|_{L^2(\Omega, E)}.$$

If  $E$  does not contain a closed subspace isomorphic to  $c_0$ , then by a result of Hoffmann-Jørgensen and Kwapień (cf. [10, Theorem 9.29]),  $\gamma(\mathcal{H}, E) = \gamma_\infty(\mathcal{H}, E)$ .

We will apply the above notions to the space  $\mathcal{H} = L^2(\mathbb{R}_+, H)$  where  $H$  is a separable Hilbert space. For an operator-valued function  $\phi : \mathbb{R}_+ \rightarrow \mathcal{B}(H, E)$  which is  *$H$ -strongly measurable* in the sense that  $t \mapsto \phi(t)h$  is strongly measurable for all  $h \in H$ , and *weakly square integrable* in the sense that  $t \mapsto \phi^*(t)x^*$  is square Bochner integrable for all  $x^* \in E^*$ , let  $R_\phi \in \mathcal{B}(L^2(\mathbb{R}_+, H), E)$  be defined as the Pettis integral operator

$$R_\phi(f) := \int_{\mathbb{R}_+} \phi(t)f(t) dt.$$

We say that  $\phi \in \gamma(\mathbb{R}_+, H, E)$  if  $R_\phi \in \gamma(L^2(\mathbb{R}_+, H), E)$  and write

$$\|\phi\|_{\gamma(\mathbb{R}_+, H, E)} := \|R_\phi\|_{\gamma(L^2(\mathbb{R}_+, H), E)}.$$

If  $H = \mathbb{K}$ , where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  is the underlying scalar field, we write  $\gamma(\mathbb{R}_+, E)$  for  $\gamma(\mathbb{R}_+, H, E)$ . For almost summing operators we use an analogous notation.

For more information we refer to [4, 8, 11, 12].

**Hilbert and Bessel sequences.** Let  $\mathcal{H}$  be a Hilbert space and  $I \subseteq \mathbb{Z}$  an index set. A sequence  $(h_i)_{i \in I}$  in  $\mathcal{H}$  is said to be a *Hilbert sequence* if there exists a constant  $C > 0$  such that for all scalars  $(\alpha_i)_{i \in I}$ ,

$$\left( \left\| \sum_{i \in I} \alpha_i h_i \right\|^2 \right)^{1/2} \leq C \left( \sum_{i \in I} |\alpha_i|^2 \right)^{1/2}.$$

The infimum of all admissible constants  $C > 0$  will be denoted by  $C_H(\{h_i : i \in I\})$ . A Hilbert sequence that is a Schauder basis is called a *Hilbert basis* (cf. [17, Section 1.8]).

The sequence  $(h_i)_{i \in I}$  is said to be a *Bessel sequence* if there exists a constant  $c > 0$  such that for all scalars  $(\alpha_i)_{i \in I}$ ,

$$c \left( \sum_{i \in I} |\alpha_i|^2 \right)^{1/2} \leq \left( \left\| \sum_{i \in I} \alpha_i h_i \right\|^2 \right)^{1/2}.$$

The supremum of all admissible constants  $c > 0$  will be denoted by  $C_B(\{h_i : i \in I\})$ . Notice that every Bessel sequence is linearly independent. A Bessel sequence that is a Schauder basis is called a *Bessel basis*. A sequence  $(h_i)_{i \in I}$  that is a Bessel sequence and a Hilbert sequence is said to be a *Riesz sequence*. A sequence  $(h_i)_{i \in I}$  that is a Bessel basis and a Hilbert basis is said to be a *Riesz basis* (cf. [17, Section 1.8]).

In the above situation if it is clear which sequence in  $\mathcal{H}$  we refer to, we use the short-hand notation  $C_H$  and  $C_B$  for  $C_H(\{h_i : i \in I\})$  and  $C_B(\{h_i : i \in I\})$ .

In the next results we study the relation between  $\gamma$ -radonifying operators and Hilbert and Bessel sequences.

**Proposition 2.1.** *Let  $(f_n)_{n \geq 1}$  be a Hilbert sequence in  $\mathcal{H}$ .*

(a) *If  $R \in \gamma_\infty(\mathcal{H}, E)$ , then*

$$\sup_{N \geq 1} \left\| \sum_{n=1}^N \gamma_n R f_n \right\|_{L^2(\Omega, E)} \leq C_H \|R\|_{\gamma_\infty(\mathcal{H}, E)}. \quad (1)$$

(b) *If  $R \in \gamma(\mathcal{H}, E)$ , then  $\sum_{n \geq 1} \gamma_n R f_n$  converges in  $L^2(\Omega, E)$  and*

$$\left\| \sum_{n \geq 1} \gamma_n R f_n \right\|_{L^2(\Omega, E)} \leq C_H \|R\|_{\gamma(\mathcal{H}, E)}. \quad (2)$$

*Proof.* (a): Fix  $N \geq 1$  and let  $\{h_1, \dots, h_N\}$  be an orthonormal system in  $\mathcal{H}$ . Since  $(f_n)_{n \geq 1}$  is a Hilbert sequence there is a unique  $T \in \mathcal{B}(\mathcal{H})$  such that  $T h_n = f_n$  for  $n = 1, \dots, N$  and  $T x = 0$  for all  $x \in \{h_1, \dots, h_N\}^\perp$ . Moreover,  $\|T\| \leq C_H$ .

By the right ideal property we have  $R \circ T \in \gamma_\infty(\mathcal{H}, E)$  and, for all  $N \geq 1$ ,

$$\left\| \sum_{n=1}^N \gamma_n R f_n \right\|_{L^2(\Omega, E)} = \left\| \sum_{n=1}^N \gamma_n R T h_n \right\|_{L^2(\Omega, E)} \leq \|R \circ T\|_{\gamma_\infty(\mathcal{H}, E)} \leq C_H \|R\|_{\gamma_\infty(\mathcal{H}, E)}.$$

(b): This is proved in a similar way. □

**Proposition 2.2.** *Let  $(f_n)_{n \geq 1}$  be a Bessel sequence in  $\mathcal{H}$  and let  $\mathcal{H}_f$  denote its closed linear span.*

$$\begin{aligned}
\text{(a)} \quad & \text{If } \sup_{N \geq 1} \left\| \sum_{n=1}^N \gamma_n R f_n \right\|_{L^2(\Omega, E)} < \infty, \text{ then } R \in \gamma_\infty(\mathcal{H}_f, E) \text{ and} \\
& \|R\|_{\gamma_\infty(\mathcal{H}_f, E)} \leq C_B^{-1} \sup_{N \geq 1} \left\| \sum_{n=1}^N \gamma_n R f_n \right\|_{L^2(\Omega, E)}. \tag{3}
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad & \text{If } \sum_{n \geq 1} \gamma_n R f_n \text{ converges in } L^2(\Omega, E), \text{ then } R \in \gamma(\mathcal{H}_f, E) \text{ and} \\
& \|R\|_{\gamma(\mathcal{H}_f, E)} \leq C_B^{-1} \left\| \sum_{n \geq 1} \gamma_n R f_n \right\|_{L^2(\Omega, E)}. \tag{4}
\end{aligned}$$

*Proof.* Let  $(h_n)_{n \geq 1}$  an orthonormal basis for  $\mathcal{H}_f$ . Since  $(f_n)_{n \geq 1}$  is a Bessel sequence there is a unique  $T \in \mathcal{B}(\mathcal{H}, E)$  such that  $T f_n = h_n$  and  $T x = 0$  for  $x \in \mathcal{H}_f^\perp$ . Notice that  $\|T\| \leq C_B^{-1}$ . On the linear span  $\mathcal{H}_0$  of the sequence  $(f_n)_{n \geq 1}$  we define an inner product by  $[x, y]_T := [Tx, Ty]_{\mathcal{H}}$ . Note that this is well defined by the linear independence of the sequence  $(f_n)_{n \geq 1}$ . Let  $\mathcal{H}_T$  denote the Hilbert space completion of  $\mathcal{H}_0$  with respect to  $[\cdot, \cdot]_T$ . The identity mapping on  $\mathcal{H}_f$  extends to a bounded operator  $j : \mathcal{H}_f \hookrightarrow \mathcal{H}_T$  with norm  $\|j\| \leq C_B^{-1}$ . Clearly,  $(j f_n)_{n \geq 1}$  is an orthonormal sequence in  $\mathcal{H}_T$  with dense span, and therefore it is an orthonormal basis for  $\mathcal{H}_T$ . It is elementary to verify that the assumption on  $R$  may now be translated as saying that  $R$  extends in a unique way to an almost summing operator (in part (a)), respectively a  $\gamma$ -radonifying operator (in part (b)), denoted by  $R_T$ , from  $\mathcal{H}_T$  to  $E$ . We estimate

$$\left\| \sum_{n \geq 1} \alpha_n j h_n \right\|_{\mathcal{H}_T} = \left\| \sum_{n \geq 1} \alpha_n T h_n \right\|_{\mathcal{H}} \leq C_B^{-1} \left\| \sum_{n \geq 1} \alpha_n h_n \right\|_{\mathcal{H}} = C_B^{-1} \left( \sum_{n \geq 1} |\alpha_n|^2 \right)^{1/2}.$$

From this we deduce that  $(j h_n)_{n \geq 1}$  is a Hilbert sequence in  $\mathcal{H}_T$  with constant  $\leq C_B^{-1}$ . Hence we may apply Proposition 2.1 to the operator  $R_T : \mathcal{H}_T \rightarrow E$  and the Hilbert sequence  $(j h_n)_{n \geq 1}$  in  $\mathcal{H}_T$  to obtain the result.  $\square$

As a consequence of the above results we obtain:

**Theorem 2.3.** *Let  $(f_n)_{n \geq 1}$  be a Riesz basis in the Hilbert space  $\mathcal{H}$ .*

- (a) *One has  $R \in \gamma_\infty(\mathcal{H}, E)$  if and only if  $\sup_{N \geq 1} \left\| \sum_{n=1}^N \gamma_n R f_n \right\|_{L^2(\Omega, E)} < \infty$ . In that case (1) and (3) hold.*
- (b) *One has  $R \in \gamma(\mathcal{H}, E)$  if and only if  $\sum_{n \geq 1} \gamma_n R f_n$  converges in  $L^2(\Omega, E)$ . In that case (2) and (4) hold.*

The following well-known lemma identifies a class of Riesz sequences in  $L^2(\mathbb{R})$ . For convenience we include the short proof from [1, Theorem 2.1]. Let  $\mathbb{T}$  be the unit circle in  $\mathbb{C}$ .

**Lemma 2.4.** *Let  $f \in L^2(\mathbb{R})$  and define the sequence  $(f_n)_{n \in \mathbb{Z}}$  in  $L^2(\mathbb{R})$  by  $f_n(t) = e^{2\pi n i t} f(t)$ . Define  $F : \mathbb{T} \rightarrow \mathbb{R}$  as*

$$F(e^{2\pi i t}) := \sum_{k \in \mathbb{Z}} |f(t + k)|^2$$

- (a) *The sequence  $(f_n)_{n \in \mathbb{Z}}$  is a Bessel sequence in  $L^2(\mathbb{R})$  if and only if there exists a constant  $A > 0$  such that  $A \leq F(e^{2\pi i t})$  for almost all  $t \in [0, 1]$ .*
- (b) *The sequence  $(f_n)_{n \in \mathbb{Z}}$  is a Hilbert sequence in  $L^2(\mathbb{R})$  if and only if there exists a constant  $B > 0$  such that  $F(e^{2\pi i t}) \leq B$  for almost all  $t \in [0, 1]$ .*

*In these cases,  $C_B^2 = \text{ess inf } F$  and  $C_H^2 = \text{ess sup } F$  respectively.*

*Proof.* Both assertions are obtained by observing that for  $I \subseteq \mathbb{Z}$  and  $(a_n)_{n \in I}$  in  $\mathbb{C}$  we may write

$$\begin{aligned} \left\| \sum_{n \in I} a_n f_n \right\|_{L^2(\mathbb{R})}^2 &= \sum_{k \in \mathbb{Z}} \int_k^{(k+1)} \left| \sum_{n \in I} a_n e^{2\pi n i t} f(t) \right|^2 dt \\ &= \sum_{k \in \mathbb{Z}} \int_0^1 \left| \sum_{n \in I} a_n e^{2\pi n i t} f(t+k) \right|^2 dt = \int_0^1 \left| \sum_{n \in I} a_n e^{2\pi n i t} \right|^2 F(e^{2\pi i t}) dt. \end{aligned}$$

□

The following application of Lemma 2.4 will be used below.

*Example 2.5.* Let  $\rho \in [0, 1)$  and  $a > 0$ . For  $n \in \mathbb{Z}$  let

$$f_n(t) = e^{-at+2\pi(n+\rho)it} \mathbb{1}_{[0,\infty)}(t).$$

Then  $(f_n)_{n \in \mathbb{Z}}$  is a Riesz sequence in  $L^2(\mathbb{R})$  with constants  $C_B^2 = \frac{e^{-2a}}{e^{2a}-1}$  and  $C_H^2 = \frac{e^{2a}}{e^{2a}-1}$ . Indeed, let  $f(t) := e^{-at+2\pi\rho it} \mathbb{1}_{[0,\infty)}(t)$ . For all  $t \in [0, 1)$ ,

$$F(e^{2\pi i t}) = \sum_{k \in \mathbb{Z}} |f(t+k)|^2 = \sum_{k=0}^{\infty} e^{-2a(t+k)} = \frac{e^{2a(1-t)}}{e^{2a}-1}.$$

Now Lemma 2.4 implies the result.

*Remark 2.6.* Necessary and sufficient conditions on the complex coefficients  $c_n$  and  $\lambda_n$  with  $\operatorname{Re} \lambda_n > 0$  in order that the functions  $z \mapsto c_n \exp(-\lambda_n z)$  form a Riesz sequence can be found in [13, Section 10.3] and [7].

### 3. MAIN RESULTS

In this section we use Proposition 2.1 to obtain an alternative proof of [12, Theorem 3.4] on the  $R$ -boundedness of certain Laplace transforms. This result is applied to strongly continuous semigroups to obtain estimates for the abscissa of  $R$ -boundedness of the resolvent. From this we deduce Theorem 1.1a as well as bounded perturbation results for the existence of solutions and invariant measures for the problem  $(\text{SCP})_{(A,B)}$ .

Let  $(r_n)_{n \geq 1}$  be a Rademacher sequence on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . A family of operators  $\mathcal{T} \subseteq \mathcal{B}(E)$  is called  $R$ -bounded if there exists a constant  $C > 0$  such that for all  $N \geq 1$  and all sequences  $(T_n)_{n=1}^N \subseteq \mathcal{T}$  and  $(x_n)_{n=1}^N \subseteq E$  we have

$$\mathbb{E} \left\| \sum_{n=1}^N r_n T_n x_n \right\|^2 \leq C^2 \mathbb{E} \left\| \sum_{n=1}^N r_n x_n \right\|^2.$$

The least possible constant  $C$  is called the  $R$ -bound of  $\mathcal{T}$ , notation  $\mathcal{R}(\mathcal{T})$ . Clearly, every  $R$ -bounded family  $\mathcal{T}$  is uniformly bounded and  $\sup_{T \in \mathcal{T}} \|T\| \leq \mathcal{R}(\mathcal{T})$ .

Following [12], for an operator  $T \in \mathcal{B}(L^2(\mathbb{R}_+), E)$  we define the *Laplace transform*  $\hat{T} : \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\} \rightarrow E$  as

$$\hat{T}(\lambda) := T e_\lambda.$$

Here  $e_\lambda \in L^2(\mathbb{R}_+)$  is given by  $e_\lambda(t) = e^{-\lambda t}$ . For a Banach space  $F$  and a bounded operator  $\Theta : F \rightarrow \mathcal{B}(L^2(\mathbb{R}_+), E)$  we define the *Laplace transform*  $\hat{\Theta} : \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\} \rightarrow \mathcal{B}(F, E)$  as

$$\hat{\Theta}(\lambda)y := \widehat{\Theta y}(\lambda) \quad \operatorname{Re} \lambda > 0, y \in F.$$

The following result is a slight refinement of [12, Theorem 3.4]. The main novelty is the simple proof of the estimate (5).

**Theorem 3.1.** *Let  $F$  be a Banach space. Let  $\Theta : F \rightarrow \gamma_\infty(L^2(\mathbb{R}_+), E)$  be a bounded operator and let  $\delta > 0$ . Then  $\widehat{\Theta}$  is  $R$ -bounded on the half-plane  $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \delta\}$  and there exists a universal constant  $C$  such that*

$$\mathcal{R}(\{\widehat{\Theta}(\lambda) : \operatorname{Re} \lambda \geq \delta\}) \leq \|\Theta\| \frac{C}{\sqrt{\delta}}.$$

*Proof.* Let  $\delta > 0$ . Consider the set  $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda = \delta\}$ . Fix  $\sigma \in [\delta/2, 3\delta/2]$  and  $\rho \in [0, 1)$ . For  $n \in \mathbb{Z}$  let  $g_n : \mathbb{R}_+ \rightarrow \mathbb{C}$  be given by

$$g_n(t) = e^{-\sigma t + (n+\rho)\delta i t}.$$

By substitution, this reduces to Example 2.5, whence  $(g_n)_{n \geq 1}$  is a Riesz sequence in  $L^2(\mathbb{R}_+)$  with constant  $0 < C_H \leq \left(\frac{C}{\delta}\right)^{1/2}$  where  $C := 2\pi \frac{e^{2\pi}}{e^{2\pi}-1}$ . For  $y \in F$ , we may apply Proposition 2.1 to obtain

$$\begin{aligned} \left\| \sum_{n=-N}^N \gamma_n \widehat{\Theta}(\sigma - (n+\rho)\delta i) y \right\|_{L^2(\Omega, E)} &= \left\| \sum_{n=-N}^N \gamma_n (\Theta y) g_n \right\|_{L^2(\Omega, E)} \\ &\leq C_H \|\Theta y\|_{\gamma_\infty(\Omega, E)} \leq \left(\frac{C}{\delta}\right)^{1/2} \|\Theta\| \|y\|. \end{aligned} \quad (5)$$

The rest of the proof follows the lines in [12].  $\square$

In what follows we let  $(T(t))_{t \geq 0}$  be a strongly continuous semigroup on  $E$  with generator  $A$ . We recall from [11, 12] that the problem  $(\text{SCP})_{(A, B)}$  admits a (unique) solution if and only if  $T(\cdot)B$  belongs to  $\gamma([0, T], H, E)$  for some (all)  $T > 0$ . Furthermore, an invariant measure exists if and only if  $T(\cdot)B$  belongs to  $\gamma(\mathbb{R}_+, H, E)$ .

The next theorem improves [12, Theorem 1.3], where the bound  $s_R(A) \leq 0$  was obtained.

**Theorem 3.2.** *Assume that for all  $x \in E$ ,  $T(\cdot)x \in \gamma_\infty(\mathbb{R}_+, E)$ . Then  $s_R(A) < 0$ , i.e., there exists an  $\varepsilon > 0$  such that  $\{R(\lambda, A) : \operatorname{Re} \lambda \geq -\varepsilon\}$  is  $R$ -bounded.*

*Proof.* By the closed graph theorem there exists an  $M > 0$  such that  $\|T(\cdot)x\|_{\gamma_\infty(\mathbb{R}_+, E)} \leq M\|x\|$ . By Theorem 3.1,  $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\} \subseteq \varrho(A)$  and

$$\mathcal{R}(\{R(\lambda, A) : \operatorname{Re} \lambda \geq \delta\}) \leq \frac{c}{\sqrt{\delta}} \quad (6)$$

for all  $\delta > 0$ , where  $c := CM$  with  $C$  the universal constant of Theorem 3.1. The following standard argument shows that this implies the bound

$$s(A) \leq -\frac{1}{4c^2}. \quad (7)$$

Choose  $\delta > 0$  and let  $\mu \in \sigma(A)$  be such that  $\operatorname{Re} \mu > s(A) - \delta$ . With  $\lambda = \frac{1}{4c^2} + i \operatorname{Im} \mu$  it follows that

$$\frac{1}{4c^2} - s(A) + \delta \geq \operatorname{dist}(\lambda, \sigma(A)) \geq \frac{1}{\|R(\lambda, A)\|} \geq \frac{\sqrt{\operatorname{Re} \lambda}}{c} = \frac{1}{2c^2}.$$

Thus  $s(A) \leq -\frac{1}{4c^2} + \delta$ . Since  $\delta > 0$  was arbitrary, this gives (7).

Now let  $\varepsilon_0 := \frac{1}{4c^2}$ . For  $\lambda$  with  $-\varepsilon_0 < \operatorname{Re} \lambda < 3\varepsilon_0$  we may write

$$R(\lambda, A) = \sum_{n \geq 0} (\varepsilon_0 - \operatorname{Re} \lambda)^n R(\varepsilon_0 + i \operatorname{Im} \lambda, A)^{n+1}.$$

Fix  $0 < \varepsilon < \varepsilon_0$ . We claim that  $\{R(\lambda, A) : \operatorname{Re} \lambda = -\varepsilon\}$  is  $R$ -bounded. To see this let  $(r_k)_{k=1}^K$  be a Rademacher sequence on  $(\Omega, \mathcal{F}, \mathbb{P})$ , let  $(\lambda_k)_{k=1}^K$  be such that

$\operatorname{Re}\lambda_k = -\varepsilon$ , and let  $(x_k)_{k=1}^K$  be a sequence in  $E$ . We may estimate

$$\begin{aligned}
\left\| \sum_{k=1}^K r_k R(\lambda_k, A) x_k \right\|_{L^2(\Omega, E)} &= \left\| \sum_{n \geq 0} \sum_{k=1}^K r_k (\varepsilon_0 + \varepsilon)^n R(\varepsilon_0 + i\operatorname{Im}\lambda_k, A)^{n+1} x_k \right\|_{L^2(\Omega, E)} \\
&\leq \sum_{n \geq 0} (\varepsilon_0 + \varepsilon)^n \left\| \sum_{k=1}^K r_k R(\varepsilon_0 + i\operatorname{Im}\lambda_k, A)^{n+1} x_k \right\|_{L^2(\Omega, E)} \\
&\leq \sum_{n \geq 0} (\varepsilon_0 + \varepsilon)^n \left( \frac{c}{\sqrt{\varepsilon_0}} \right)^{n+1} \left\| \sum_{k=1}^K r_k x_k \right\|_{L^2(\Omega, E)} \\
&= \frac{1}{\varepsilon_0 - \varepsilon} \left\| \sum_{k=1}^K r_k x_k \right\|_{L^2(\Omega, E)},
\end{aligned}$$

where we used that  $\varepsilon_0 = 1/4c^2$ . This proves the claim. Now the result is obtained via [16, Proposition 2.8].  $\square$

As an application of Theorem 3.2 we have the following bounded perturbation result for the existence of a solution for the perturbed problem.

**Theorem 3.3.** *Let  $P \in \mathcal{B}(E)$  and  $B \in \mathcal{B}(H, E)$ . If  $(\operatorname{SCP})_{(A, B)}$  has a solution, then  $(\operatorname{SCP})_{(A+P, B)}$  has a solution as well.*

*Proof.* For  $\omega \in \mathbb{R}$  denote  $A_\omega = A - \omega$  and  $T_\omega(\cdot) := e^{-\omega \cdot} T(\cdot)$ . It follows from [12, Proposition 4.5] that for all  $\omega > \omega_0(A)$ ,  $T_\omega(\cdot)B \in \gamma(\mathbb{R}_+, H, E)$ . From [9, Corollary 2.17] it follows that for all  $\omega > \omega_0(A) + 1$ ,

$$\mathcal{R}(\{R(\lambda, A_\omega) : \operatorname{Re}\lambda \geq 0\}) \leq \frac{c}{\omega - \omega_0(A) - 1},$$

where  $c$  is a constant depending only on  $(T(t))_{t \geq 0}$ . Choose  $\omega_1 > \omega_0(A) + 1$  so large that  $\frac{c}{\omega_1 - \omega_0(A) - 1} \|P\| < 1$ . By [12, Lemma 5.1],  $R(i\cdot, A_{\omega_1})B \in \gamma(\mathbb{R}_+, H, E)$ .

Denote by  $(S(t))_{t \geq 0}$  the semigroup generated by  $A+P$  (cf. [5, Section III.1] or [15, Chapter III]) and let  $S_{\omega_1}(t) := e^{-\omega_1 t} S(t)$ ,  $t \geq 0$ . Since

$$\mathcal{R}(\{R(is, A_{\omega_1})P : s \in \mathbb{R}\}) \leq \mathcal{R}(\{R(is, A_{\omega_1}) : s \in \mathbb{R}\}) \|P\| =: C < 1,$$

it follows from  $i\mathbb{R} \subseteq \varrho(A_{\omega_1})$  that  $i\mathbb{R} \subseteq \varrho(A_{\omega_1} + P)$  and

$$R(is, A_{\omega_1} + P)B = \sum_{n=0}^{\infty} (R(is, A_{\omega_1})P)^n R(is, A_{\omega_1})B =: R_{A, P, \omega_1}(s) R(is, A_{\omega_1})B.$$

Moreover, as in Theorem 3.2, and using the fact that  $C < 1$ ,  $\{R_{A, P, \omega_1}(s) : s \in \mathbb{R}\}$  is  $R$ -bounded with constant  $\frac{1}{1-C}$ . From [8, Proposition 4.11] we deduce that

$$\|R(i\cdot, A_{\omega_1} + P)B\|_{\gamma(\mathbb{R}, H, E)} \leq \frac{1}{1-C} \|R(i\cdot, A_{\omega_1})B\|_{\gamma(\mathbb{R}, H, E)}.$$

Now [12, Lemma 5.1] shows that  $S_{\omega_1}(\cdot)B \in \gamma(\mathbb{R}_+, H, E)$ . It follows from the right ideal property that for all  $t > 0$ ,

$$\|S(\cdot)B\|_{\gamma(0, t, H, E)} \leq e^{t\omega_1} \|S_{\omega_1}(\cdot)B\|_{\gamma(0, t, H, E)}$$

and the result can be obtained via [11, Theorem 7.1].  $\square$

Concerning existence and uniqueness of invariant measures we obtain:

**Theorem 3.4.** *Assume that  $s(A) < 0$  and that  $\{R(is, A) : s \in \mathbb{R}\}$  is  $R$ -bounded. Let  $B \in \mathcal{B}(H, E)$  such that  $(\operatorname{SCP})_{(A, B)}$  admits an invariant measure. Then there exists a  $\delta > 0$  such that for all  $P \in \mathcal{B}(E)$  with  $\|P\| < \delta$ ,  $(\operatorname{SCP})_{(A+P, B)}$  admits a unique invariant measure.*



*Proof.* Let  $\delta > 0$  such that  $\mathcal{R}(\{R(is, A) : s \in \mathbb{R}\}) \leq 1/\delta$ . Then, if  $\|P\| < \delta$ ,

$$\mathcal{R}(\{R(is, A)P : s \in \mathbb{R}\}) \leq \mathcal{R}(\{R(is, A) : s \in \mathbb{R}\})\|P\| =: C < 1.$$

As in Theorem 3.3 it can be deduced that

$$\|R(i\cdot, A+P)B\|_{\gamma(\mathbb{R}, H, E)} \leq \frac{1}{1-C} \|R(i\cdot, A)B\|_{\gamma(\mathbb{R}, H, E)}.$$

The existence of an invariant measure now follows from [12, Proposition 4.4 and Lemma 5.1].

By [12, Corollary 4.3], for uniqueness it suffices to note that  $R(\lambda, A+P)$  is uniformly bounded for  $\operatorname{Re} \lambda > 0$ .  $\square$

In particular, the  $R$ -boundedness of  $\{R(is, A) : s \in \mathbb{R}\}$  implies that an invariant measure for  $(\operatorname{SCP})_{(A, B)}$ , if one exists, is unique. On the other hand, if  $i\mathbb{R} \subseteq \varrho(A)$  but  $\{R(is, A) : s \in \mathbb{R}\}$  fails to be  $R$ -bounded, then Theorem 3.2 shows that there exists a rank one operator  $B' \in \mathcal{B}(H, E)$  such that the problem  $(\operatorname{SCP})_{(A, B')}$  fails to have an invariant measure. As a result we obtain that if  $(\operatorname{SCP})_{(A, B)}$  fails to have a unique invariant measure, then there exists a rank one operator  $B' \in \mathcal{B}(H, E)$  such that the problem  $(\operatorname{SCP})_{(A, B')}$  fails to have an invariant measure. A related result can be found in [6].

*Proof of Theorems 1.1a and 1.1b.* If  $T(\cdot)x \in \gamma(\mathbb{R}_+, E)$  for all  $x \in E$ , then by Theorem 3.2  $s(A) < 0$  and  $\{R(is, A) : s \in \mathbb{R}\}$  is  $R$ -bounded. Thus,

Theorem 3.4 applies to the bounded perturbation  $P = \delta \cdot I_E$ .  $\square$

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